

Features of the fractional diffusion-advection equation

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Abstract

We advance an exact, explicit form for the solutions to the fractional diffusion-advection equation. Numerical analysis of this equation shows that its solutions resemble power-laws.

Keywords: distributions; diffusion-advection; power-laws

1 Introduction

Advection constitutes a transport mechanism of either a substance or a conserved property by a fluid. This process originates in the fluid's bulk motion. For instance, the transport of silt in a river by bulk water flowing downstream. Thermodynamic advected quantities are energy and enthalpy. Any substance or conserved, extensive quantity can experience advection by a fluid that contains it. Advection does not include transport by molecular diffusion. In this work we analyze some features of an evolution equation characterizing the combined effects of i) advection and ii) a diffusion process described by fractional partial derivatives. The ensuing fractional advection-diffusion equation has been recently applied to the study of the transport of energetic particles in the interplanetary environment [1, 2, 3, 5, 6].

There is a considerable body of evidence, from data collected by spacecrafts like *Ulysses* and *Voyager 2*, indicating that the transport of energetic particles in the turbulent heliospheric medium is superdiffusive [7, 8]. Considerable effort has been devoted in recent years to the development of superdiffusive models for the transport of electrons and protons in the heliosphere [1, 2, 3]. This kind of transport regime exhibits a power-law growth of the mean square displacement of the diffusing particles, $\langle \Delta x^2 \rangle \propto t^\alpha$, with $\alpha > 1$ (see, for instance, [4]). The special case $\alpha = 2$ is called ballistic transport. The limit case $\alpha \rightarrow 1$ corresponds to normal diffusion, described by the well-known Gaussian propagator. The energetic particles detected by the aforementioned probes are usually associated with violent solar events like solar flares. These particles diffuse in the solar wind, which is a turbulent environment than can be assumed statistically homogeneous at large enough distances from the sun [7]. This implies that the propagator $P(x, x', t, t')$, describing the probability of finding at the space time location (x, t) a particle that has been injected at (x', t') , depends solely on the differences $x - x'$ and $t - t'$. In the superdiffusive regime the propagator $P(x, x', t, t')$ is not Gaussian, and exhibits power-law tails. It arises as solution a non local diffusive process governed by an integral equation that can be recast under the guise of a diffusion equation where the well-known Laplacian term is replaced by a term involving fractional derivatives [5]. Diffusion equations with fractional derivatives have attracted considerable attention recently (see [9, 10, 11, 12, 13] and references therein) and have lots of potential applications [14, 15]. In particular, the observed distributions of solar cosmic ray particles are often consistent with power-law tails, suggesting that a superdif-

fusive process is at work.

A proper understanding of the transport of energetic particles in space is a vital ingredient for the analysis of various important phenomena, such as the propagation of particles from the Sun to our planet or, more generally, the acceleration and transport of cosmic rays. The superdiffusion of particles in interplanetary turbulent environments is often modelled using asymptotic expressions for the pertinent non-Gaussian propagator, which have a limited range of validity. A first step towards a more accurate analytical treatment of this problem was recently provided by Litvinenko and Effenberger (LE) in [6]. LE considered solutions of a fractional diffusion-advection equation describing the diffusion of particles emitted at a shock front that propagates at a constant upstream speed V_{sh} in the solar wind rest frame. The shock front is assumed to be planar, leading to an effectively one-dimensional problem. Each physical quantity depends only on the time t and on the spatial coordinate x measured along an axis perpendicular to the shock front.

In the present contribution we re-visit the fractional diffusion-advection equation and provide **explicit**, exact closed analytical solutions, without approximations. We also undertake a numerical analysis that shows the these solutions resemble power-laws.

2 Formulation of the Problem

We deal with the equation

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^\alpha f}{\partial |x|^\alpha} + a \frac{\partial f}{\partial x} + \delta(x), \quad (2.1)$$

where $t > 0$ and $f(x, t)$ is the distribution function for solar cosmic-rays transport. Here the fractional spatial derivative is defined as [6]

$$\frac{\partial^\alpha f}{\partial |x|^\alpha} = \frac{1}{\pi} \sin\left(\frac{\pi\alpha}{2}\right) \Gamma(\alpha + 1) \int_0^\infty \frac{f(x + \xi) - 2f(x) + f(x - \xi)}{\xi^{\alpha+1}} d\xi. \quad (2.2)$$

To solve this equation one appeals to the the Green function governed by the equation:

$$\frac{\partial \mathcal{G}}{\partial t} = \kappa \frac{\partial^\alpha \mathcal{G}}{\partial |x|^\alpha} + \delta(x)\delta(t). \quad (2.3)$$

With this Green function, the solution of (2.1) can be expressed as

$$f(x, t) = \int_0^t \mathcal{G}(x + at', t') dt'. \quad (2.4)$$

In this work we obtain the solutions of Eqs. (2.1) and (2.3) **using distributions as main tools** [16].

For our task we use, as a first step, the solution obtained in [6] for the Green function through the use of the Fourier Transform given by

$$\hat{\mathcal{G}}(k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{G}(x, t) e^{-ikx} dx, \quad (2.5)$$

from which we obtain for $\hat{\mathcal{G}}$:

$$\hat{\mathcal{G}}(k, t) = -\kappa |k|^\alpha \hat{\mathcal{G}}(k, t) + \frac{1}{2\pi} \delta(t), \quad (2.6)$$

whose solution is

$$\hat{\mathcal{G}}(k, t) = \frac{H(t)}{2\pi} e^{-\kappa |k|^\alpha t}, \quad (2.7)$$

where $H(t)$ is the Heaviside's step function.

3 Explicit general solution of the equation

From (2.7) we have for $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}}(k, t) = \frac{H(t)}{2\pi} e^{-\kappa |k|^\alpha t} = \frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n k^{\alpha n} t^n}{n!}, \quad (3.1)$$

and, invoking the inverse Fourier transform

$$\begin{aligned} \mathcal{G}(x, t) &= \frac{H(t)}{2\pi} \int_{-\infty}^{\infty} e^{-\kappa |k|^\alpha t} e^{ikx} dk = \\ &= \frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^n}{n!} \left[\int_0^{\infty} k^{\alpha n} e^{ikx} dx + \int_0^{\infty} k^{\alpha n} e^{-ikx} dx \right]. \end{aligned} \quad (3.2)$$

Fortunately, we can find in the classical book of [16] the results for the two integrals of (3.2). We obtain

$$\mathcal{G}(x, t) = \frac{H(t)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^n}{n!} \Gamma(\alpha n + 1) \left[\frac{e^{i\frac{\pi}{2}(\alpha n + 1)}}{(x + i0)^{\alpha n + 1}} + \frac{e^{-i\frac{\pi}{2}(\alpha n + 1)}}{(x - i0)^{\alpha n + 1}} \right]. \quad (3.3)$$

Using now (2.4) we have for f

$$f(x, t) = \int_0^t \mathcal{G}(x + at', t') dt',$$

so that one can write

$$f(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n}{n!} \Gamma(\alpha n + 1) \times \int_0^t \left[\frac{e^{i\frac{\pi}{2}(\alpha n + 1)}}{(x + at' + i0)^{\alpha n + 1}} + \frac{e^{-i\frac{\pi}{2}(\alpha n + 1)}}{(x + at' - i0)^{\alpha n + 1}} \right] t'^n dt'. \quad (3.4)$$

According to Eq. (A.3) of the Appendix, where $t > 0$, we now obtain for f , invoking hypergeometric functions $F(\alpha n + 1, 2; 3; z)$ and Beta functions $\mathcal{B}(1, n + 1)$,

$$f(x, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n t^{n+1}}{n!} \Gamma(\alpha n + 1) \mathcal{B}(1, n + 1) \times \left[\frac{e^{i\frac{\pi}{2}(\alpha n + 1)}}{(x + i0)^{\alpha n + 1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{at}{x + i0}\right) + \frac{e^{-i\frac{\pi}{2}(\alpha n + 1)}}{(x - i0)^{\alpha n + 1}} F\left(\alpha n + 1, n + 1; n + 2; -\frac{at}{x - i0}\right) \right]. \quad (3.5)$$

This is the general solution of Eq.(2.1) for the initial condition $f(x, 0) = 0$.

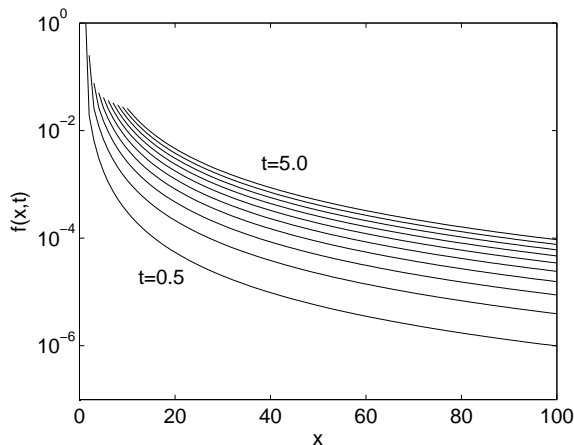


Figure 1: Typical graph for solutions of Eq. (3.5), for different times t (x in arbitrary units).

4 Results

Fig. 1 depicts a typical graph for solutions of Eq. (3.5). We do this for different times t . The main result of this contribution is that these curves are very well approximated by power laws. Figs. 2 ($\kappa = 0.01$), 3 ($\kappa = 0.5$), and 4 ($\kappa = 0.2$) show how the above solutions can be adjusted by power laws of the form $1/x^q$: the corresponding lines rotate, for fixed $\alpha = 3/2$, in the plane $f(x, t)$ vs. $1/x^q$ when at changes, as the plots clearly illustrate. In all our figures we have taken $a = 1$ and $\alpha = 3/2$.

Figs. 5, 6, and 7 show how the dependence of q with α can, in turn, be adjusted in simple fashion. In Figs. 5 and 6, straight lines are employed. In Fig. 7 we attempt a spline adjustment. It is evident that a weak diffusion-regime exists. This regime was conjectured in Ref. [6] by appeal to an approximate theoretical treatment of the fractional diffusion-advection equation and here amply confirmed by our exact approach.

What about other α values? The situation does not change. Power-law adjustment continues to be possible. Figs. 8 and 9 illustrate this issue.

5 Conclusions

We have provided an explicit analytical solution for an advection-diffusion equation involving fractional derivatives in the diffusion term. This equation governs the diffusion of particles in the solar wind injected at the front of a shock that travels at a constant upstream speed V_{sh} in the solar wind rest frame. The shock is assumed to have a planar front, leading to a problem with an effective one dimensional geometry, where all the relevant quantities depend solely on time and on the coordinate x measured along an axis perpendicular to the front.

We obtained the exact solution of the above mentioned equation in the x -configuration space (besides the associated formal solution in the k -space related to the previous one via a Fourier transform).

We conclude from our analysis that the solutions of the fractional diffusion-advection equations are essentially power laws, and have numerically found a “law” for the behavior of the associated power-exponent q with varying κ via spline interpolation.

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Appendix: Some properties of Hypergeometric Function

Using data from [17] we have

$$\int_0^t \frac{t'^n}{(x + at' \pm i\epsilon)^{\alpha n + 1}} dt' = \frac{t^{n+1}}{(x \pm i\epsilon)^{\alpha n + 1}} B(1, n + 1) \times$$

$$F\left(\alpha n + 1, n + 1, n + 2; -\frac{at}{x \pm i\epsilon}\right). \quad (\text{A.1})$$

Then, we have

$$\lim_{\epsilon \rightarrow 0^+} \int_0^t \frac{t'^n}{(x + at' \pm i\epsilon)^{\alpha n + 1}} dt' = \lim_{\epsilon \rightarrow 0^+} \frac{t^{n+1}}{(x \pm i\epsilon)^{\alpha n + 1}} B(1, n + 1) \times$$

$$\lim_{\epsilon \rightarrow 0^+} F\left(\alpha n + 1, n + 1, n + 2; -\frac{at}{x \pm i\epsilon}\right). \quad (\text{A.2})$$

Thus we obtain finally

$$\int_0^t \frac{t'^n}{(x + at' \pm i0)^{\alpha n + 1}} dt' = \frac{t^{n+1}}{(x \pm i0)^{\alpha n + 1}} B(1, n + 1) \times$$

$$F\left(\alpha n + 1, n + 1, n + 2; -\frac{at}{x \pm i0}\right). \quad (\text{A.3})$$

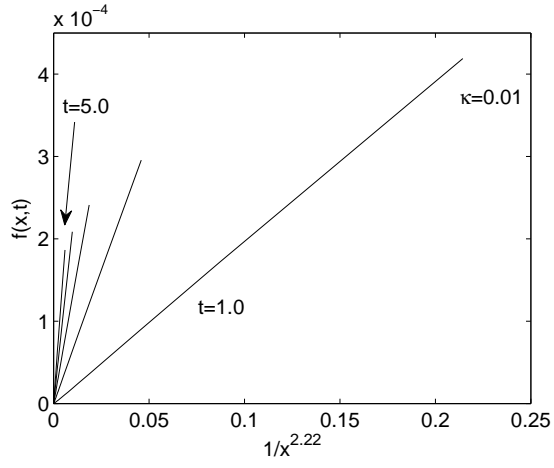


Figure 2: Power-law fitting to the solutions of Eq. (3.5) for $\alpha = 3/2$ $a = 1$, and $\kappa = 0.01$, at different times t

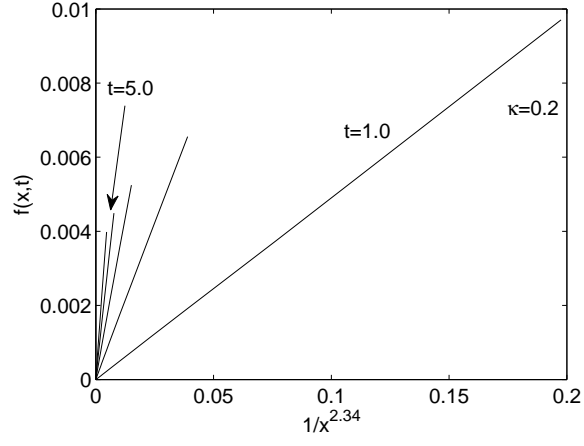


Figure 3: Power-law fitting to the solutions of Eq. (3.5) for $\alpha = 3/2$ $a = 1$, and $\kappa = 0.5$, at different times t

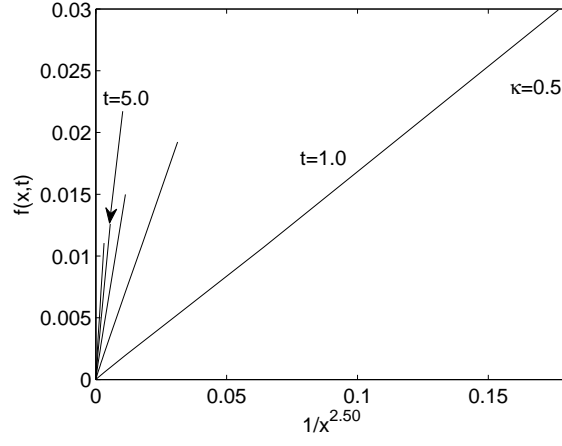


Figure 4: Power-law fitting to the solutions of Eq. (3.5) for $\alpha = 3/2$, $a = 1$, and $\kappa = 0.2$, at different times t

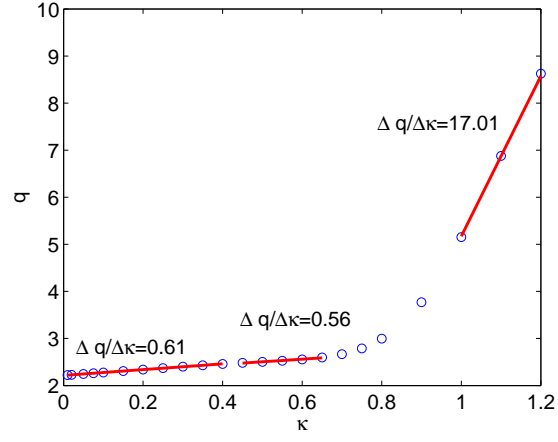


Figure 5: Possible linear relationship between the power-law exponent q and κ

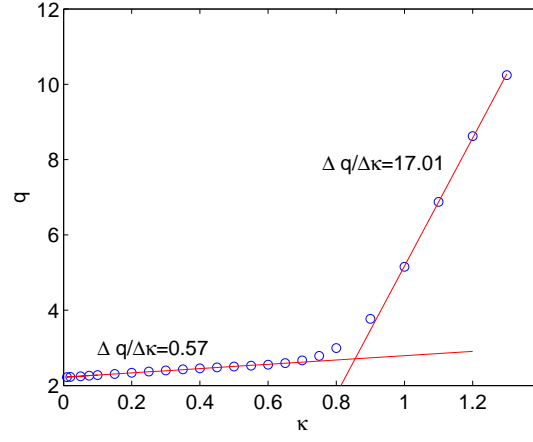


Figure 6: Possible linear relationship between the power-law exponent q and κ

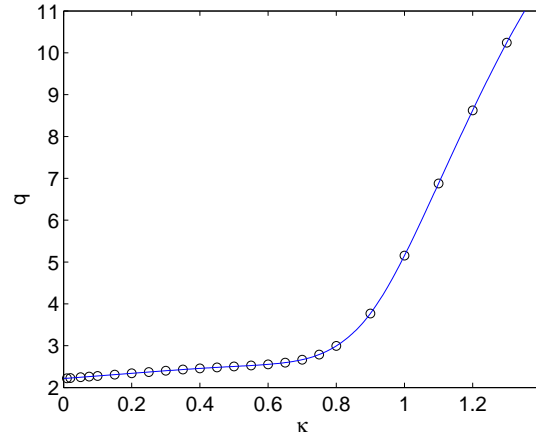


Figure 7: Spline relationship between the power-law exponent q and κ

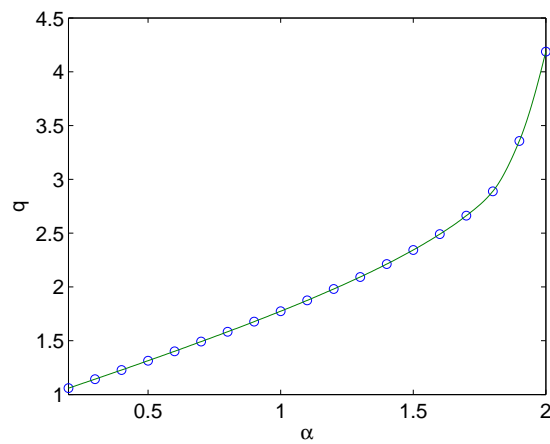


Figure 8: Fractional derivative order α vs. power law exponent q adjusting the solutions of the fractional advection-diffusion equation ($\kappa = 0.2$)

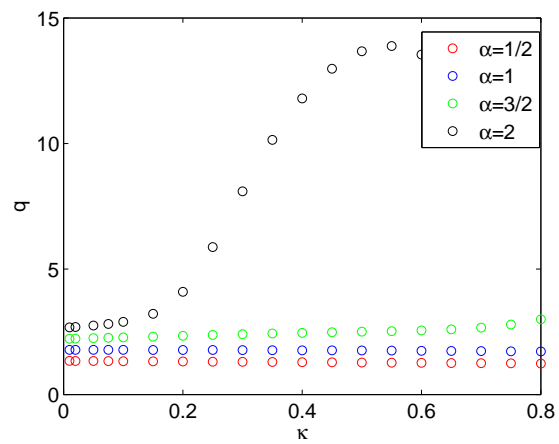


Figure 9: q vs. κ for different α -values.